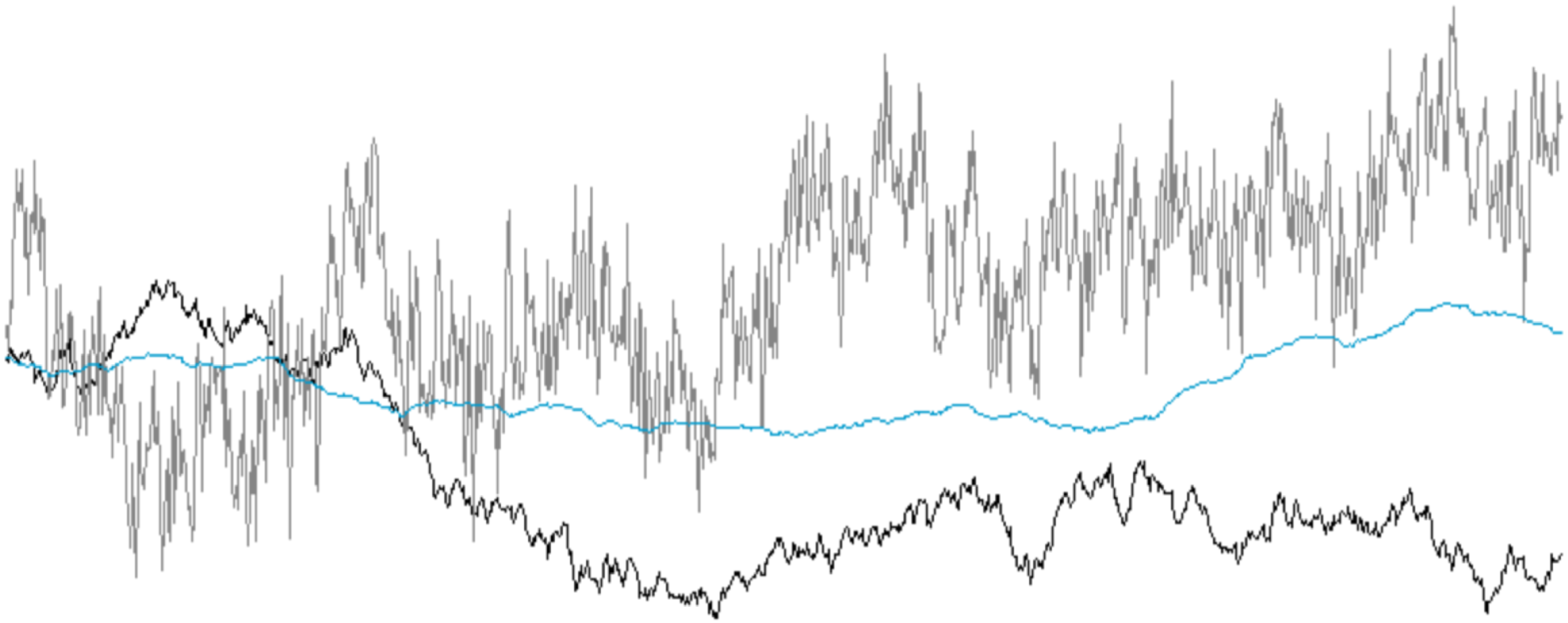
ROUGH STOCHASTIC VOLATILITY MODELING AND ITS IMPACT ON LONG-TERM LIFE INSURANCE PRICING

IA|BE PRIZE 2021 – DUPRET JEAN-LOUP



ROUGH STOCHASTIC VOLATILITY MODELING

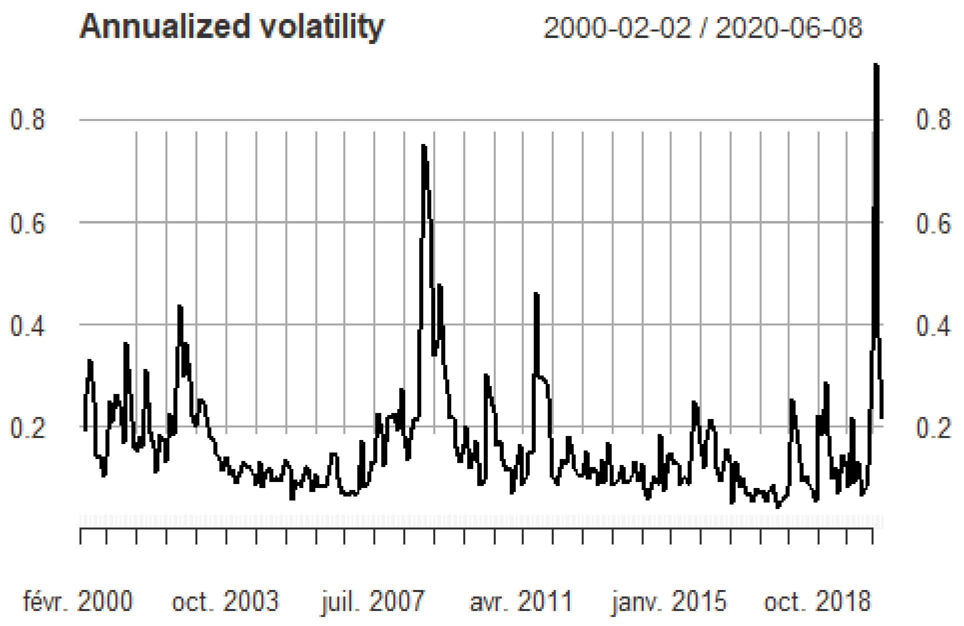
- Classical stochastic volatility models
- Fractional Brownian motion
- RFSV model
  - rBergomi
  - Rough Heston
- Calibration
- Life insurance contract

VOLATILITY IS NOT CONSTANT

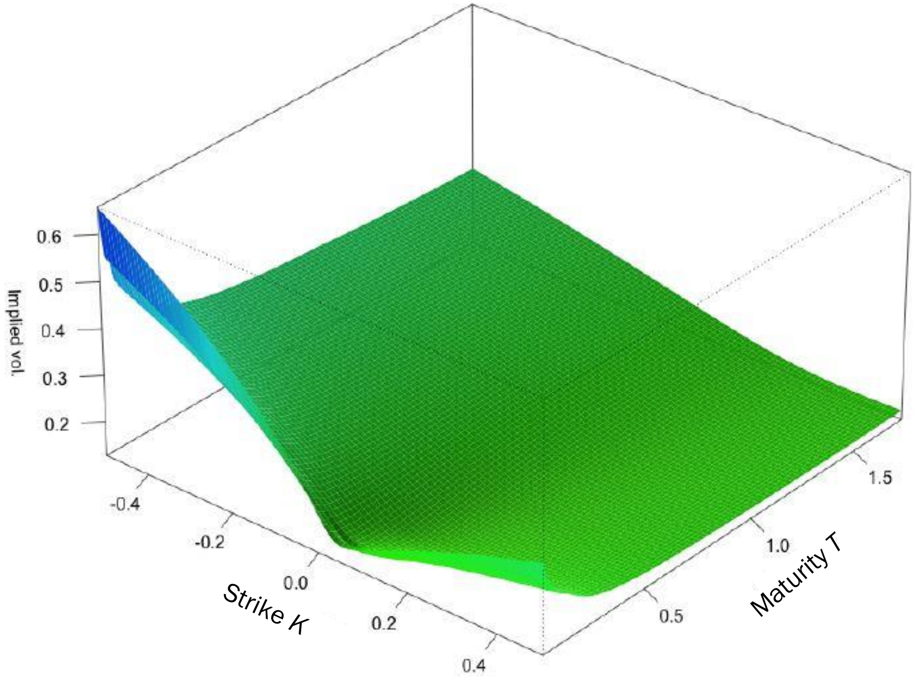
Observed volatility of stock returns is not constant but varies randomly with time :

→ **Black & Scholes** model inaccurate !

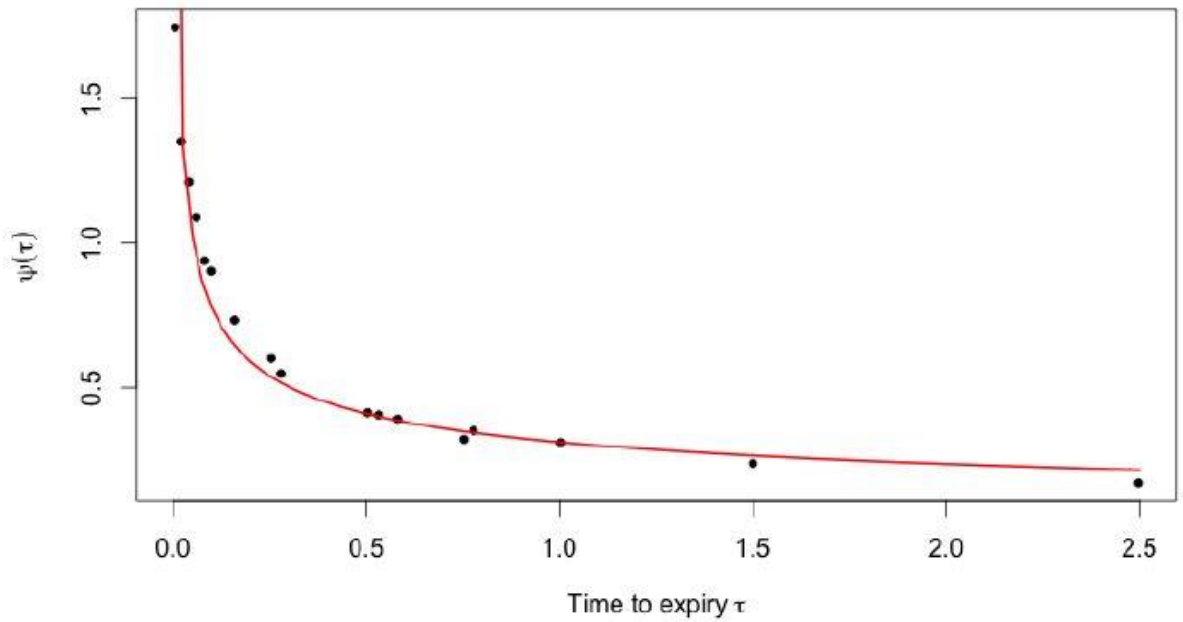
- **Historical volatility S&P500 :**



- **Implied volatility S&P500 :**



ATM VOLATILITY SKEWS : POWER-LAW



→ In red, power-law function: $\psi(\tau) = A\tau^{-0.4}$.

→ In **black**, ATM volatility skew estimated for the S&P500, 6th July

HESTON AND BATES MODELS

- Under a risk-neutral measure $\mathbb{P}^*$, the **Heston model** is given by:

$$dS_t = S_t\, r\, dt + S_t \sqrt{v_t}\, dW_t^*$$
$$dv_t = \kappa(\eta - v_t)\, dt + \theta\sqrt{v_t}\, d\hat{W}_t^*$$

with v_t the variance process and W^*, $\widehat{W}^*$ two correlated Brownian motions under $\mathbb{P}^*$.

- Under $\mathbb{P}^*$, the **Bates model** is given by:

$$\frac{dS_t}{S_t} = (r - \lambda\,\xi)\, dt + \sqrt{v_t}\, dW_t^* + (Y_t - 1)\, dN_t$$
$$dv_t = \kappa\ (\eta\ - v_t)\, dt + \theta\sqrt{v_t}\, d\hat{W}_t^*$$

with $N_t \sim Poi(\lambda\, t)$ and $\log(Y_t) \sim N(\mu_j, \sigma_j^2)$

HESTON AND BATES MODELS

ADVANTAGES :

- Incorporate **mean-reverting stochastic volatility.**
- Characteristic function in **closed-form** → fast and efficient calibration.

DRAWBACKS :

- **Implied volatility not realistic** under these two models.
- Cannot reproduce the memory properties of the **observed historical volatility.**

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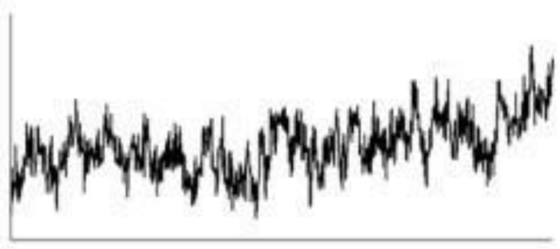
FRACTIONAL BROWNIAN MOTION (fBm)

A fractional Brownian motion $(B_t^H)_{t\geq 0}$ is a Gaussian process iif :

$$\mathrm{Cov}(B_t^H, B_s^H) = \mathbb{E}(B_t^H B_s^H) = \frac{1}{2}\{t^{2H}+s^{2H}-|t-s|^{2H}\}\,\mathbb{E}\left((B_1^H)^2\right)$$

- Depends on the parameter $H \in (0,1)$, called the **Hurst index.**
- **Stationarity** of increments.
- Increments are **positively** correlated if $H > 1/2$, **negatively** correlated if $H < 1/2$ and **independent** if $H = 1/2$:



Fractional Bm : $\boldsymbol{H < 1/2}$

Classical Bm : $\boldsymbol{H = 1/2}$

Fractional Bm : $\boldsymbol{H > 1/2}$

FRACTIONAL BROWNIAN MOTION – INCREMENTS

- The process of increments of the fBm $\Delta B_t^H = B_t^H - B_{t-1}^H$ is said to have :

  - **Long memory** for $H > 1/2$
  - **Short memory** for $H < 1/2$

→ One-to-one correspondance with the regularity of fBm trajectories.

- ΔB_t^H is called a **fractional Gaussian noise**

→ Basis of fractional mean-reverting process (**RFSV** model).

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ROUGH FRACTIONAL STOCHASTIC VOLATILITY MODEL (RFSV)

- The RFSV model is based on a fractional mean-reverting process for the **log-volatility** with $\boldsymbol{H < 1/2}$ **:**

$$dS_t = r\,S_t\,dt + \sigma_t\,S_t\,dW_t^*$$
$$dX_t = \lambda(\eta - X_t)\,dt + \nu\,dB_t^H$$

where $\boldsymbol{X_t = \log \sigma_t}$ and $\boldsymbol{dB_t^H}$ a fractional Gaussian noise with **short memory.**

- The volatility $\sigma_t = \exp(X_t)$ is the **unique stationary solution with short memory** given by :

$$\sigma_t = \exp(X_t) = \exp\left\{\eta + \int_{-\infty}^{t} e^{-\lambda(t-u)}\,dB_u^H\right\}$$

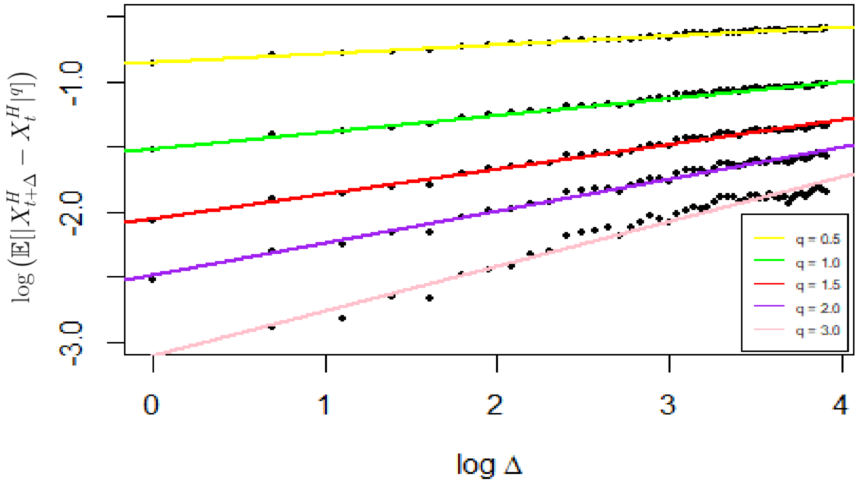
RFSV – HISTORICAL VOLATILITY (CAC40)

- Gatheral et al. (2014) show with $\lambda \to 0$:

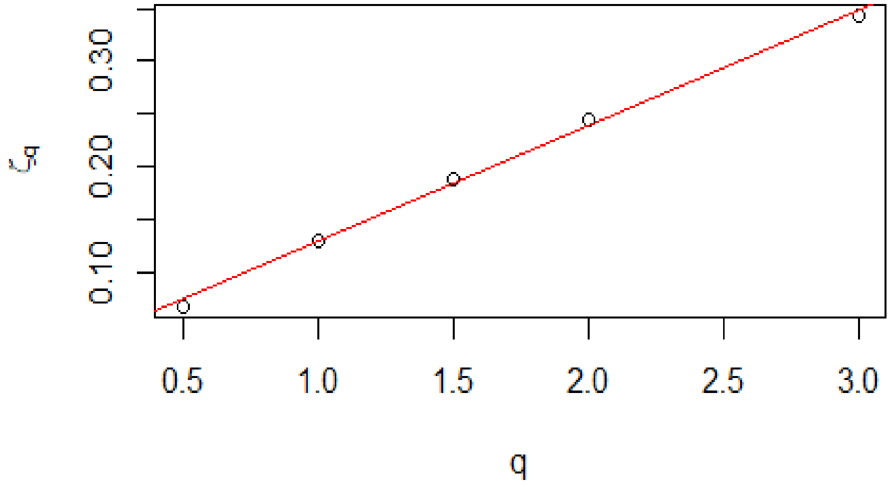
$$\mathbb{E}\Big[\sup_{t\in[0,T]} |X_t^H - X_0^H - \nu B_t^H|\Big] \to 0$$

and :

$$\mathbb{E}[|X_{t+\Delta}^H - X_t^H|^q] \to \nu^q \, K_q \, \Delta^{qH}$$



- When $\lambda \to 0$, the **log-volatility process of the RFSV** behaves as a **fBm** and approximately reproduces their **scaling property.**



→ Confirmed **empirically** with the CAC40

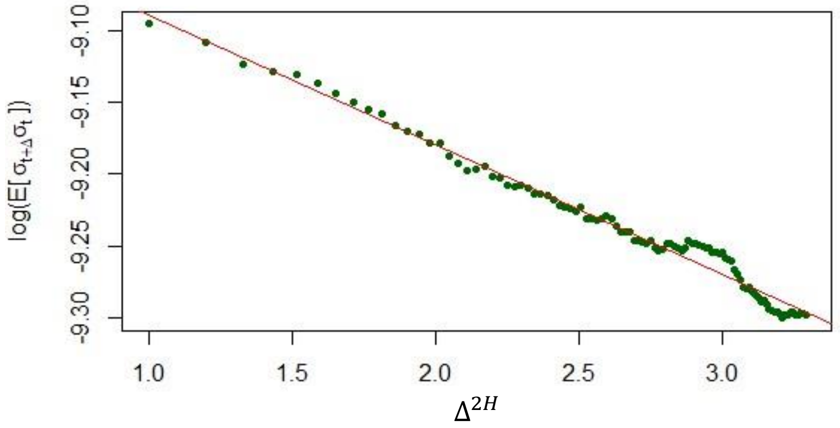
RFSV – HISTORICAL VOLATILITY (CAC40)

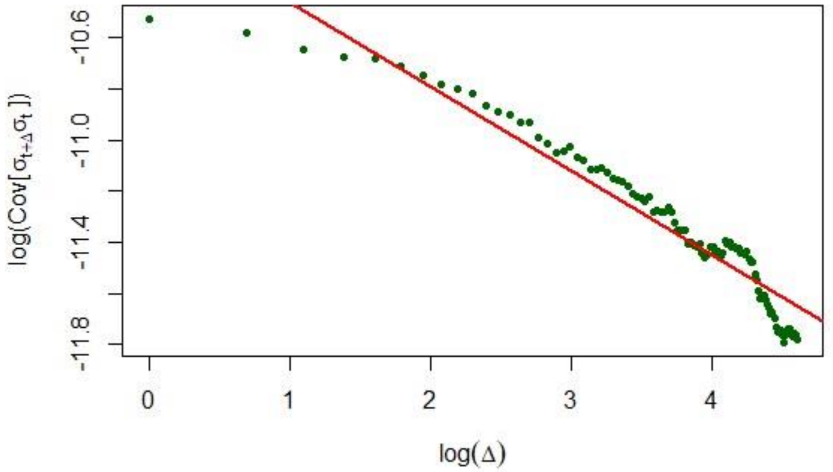
- The autocovariance of σ_t when $\lambda \to 0$ is given by :

$$\mathbb{E}[\sigma_{t+\Delta}\sigma_t] = \mathbb{E}[e^{X_t^H + X_{t+\Delta}^H}] \approx e^{2\mathbb{E}[X_t^H]+2\mathrm{Var}[X_t^H]} e^{-\nu^2 \frac{\Delta^{2H}}{2}}$$

→ $\log\left(\mathbb{E}[\sigma_{t+\Delta}\sigma_t]\right)$ is linear in Δ^{2H}, which is confirmed empirically.

→ $\mathbb{E}[\sigma_{t+\Delta}\sigma_t]$ does not behave as a **power-law function.** Nor the empirical data nor the RFSV exhibit long-term memory.



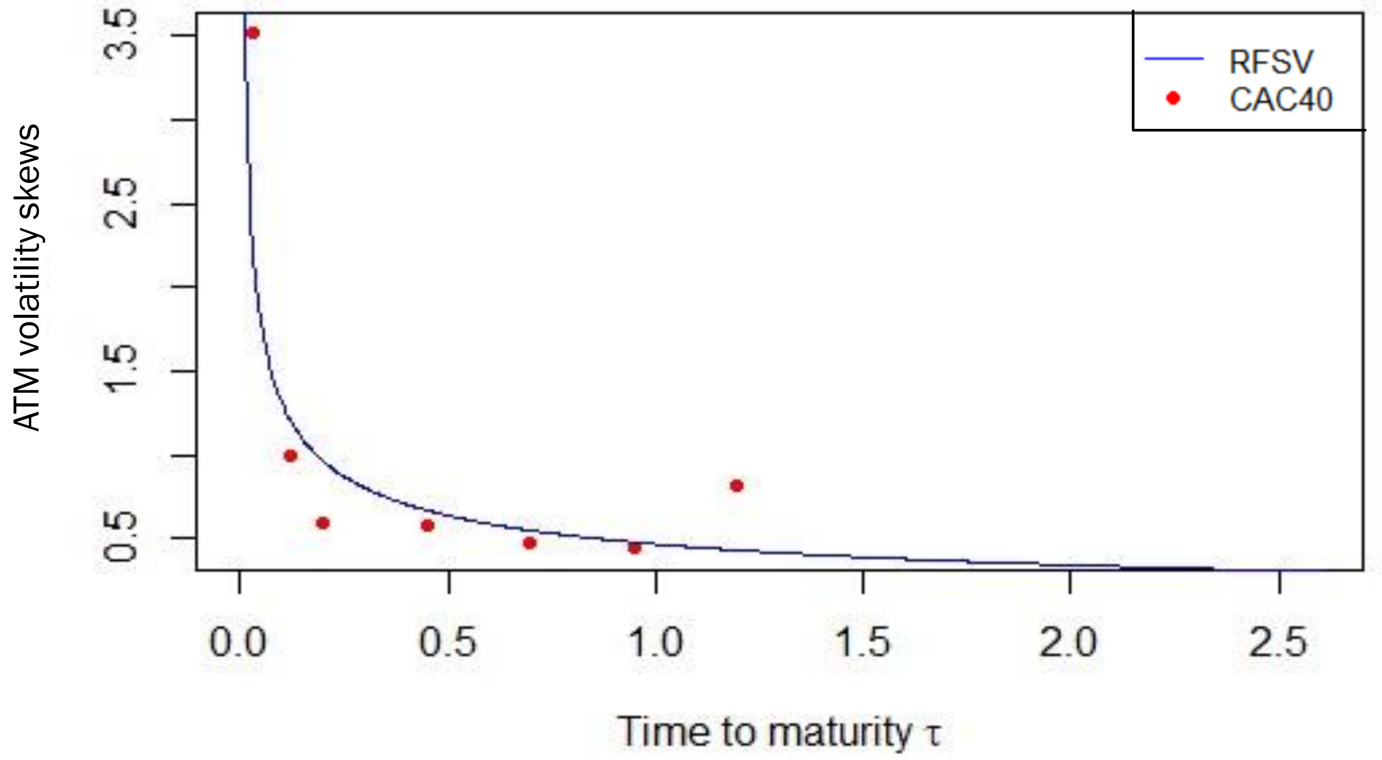


RFSV – HISTORICAL VOLATILITY

The RFSV model is extremely consistent with the **observed historical volatility** due to its short memory and rough sample paths ($H < 1/2$).

RFSV – IMPLIED VOLATILITY

→ Extremely consistent with **implied volatility** and especially with the term structure of **ATM volatility skews** :



RFSV - DRAWBACK

- BUT, the RFSV model is **too slow for pricing and calibration** since it requires a lot of slow and unstable **Monte-Carlo** simulations.

- Two more efficient models derived from the RFSV :

  → **rBergomi model**

  → **Rough Heston model**

rBERGOMI MODEL

- Model obtained from the RFSV by setting $\boldsymbol{\lambda = 0}$:

$$S_T = S_t \exp\left(r(T-t) - \frac{1}{2}\int_t^T v_u\, du + \int_t^T \sqrt{v_u}\, dW_u^{*,S} \right)$$

$$v_u = \mathbb{E}^{\mathbb{P}^*}[v_u|\mathcal{F}_t]\ \exp\left\{\eta\sqrt{2H}\int_t^u \frac{1}{(u-s)^{1/2-H}} dW_s^* - \frac{\eta^2}{2}(u-t)^{2H}\right\} = \mathbb{E}^{\mathbb{P}^*}[v_u|\mathcal{F}_t]\,\mathcal{E}\left(\eta\,\tilde{W}_t^*(u)\right)$$

- A bit more efficient and stable than the RFSV model but still **not optimal** for calibration.

- The volatility generated by the rBergomi is **not stationary** since $\lambda = 0$.

→ Inappropriate for long-term life insurance pricing

ROUGH HESTON MODEL

Extension of the **classical Heston** model with a rough fractional Gaussian noise (H < ½) :

$$dS_t = S_t\, r\, dt + S_t \sqrt{v_t}\, dW_t^*$$

$$v_t = \xi_0(t) + \frac{1}{\Gamma(H+1/2)} \int_0^t \frac{\nu}{(t-s)^{1/2-H}} \sqrt{v_s}\, d\hat{W}_s^*$$

The rough Heston is excellent for pricing long-term life insurance contracts

- **Stationary volatility** generated by the rough Heston model.
- Only 3 parameters and a characteristic function in **closed-form**

  → Pricing and calibration **far more efficient and stable.**
- Highly consistent with **historical and implied volatility.**

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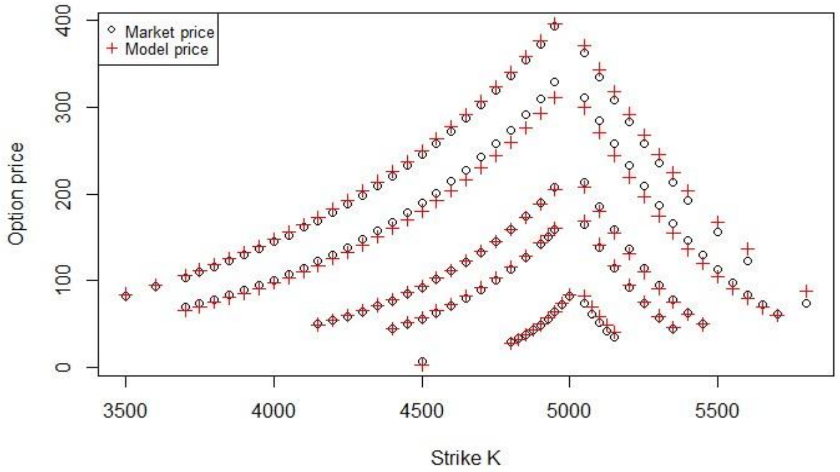
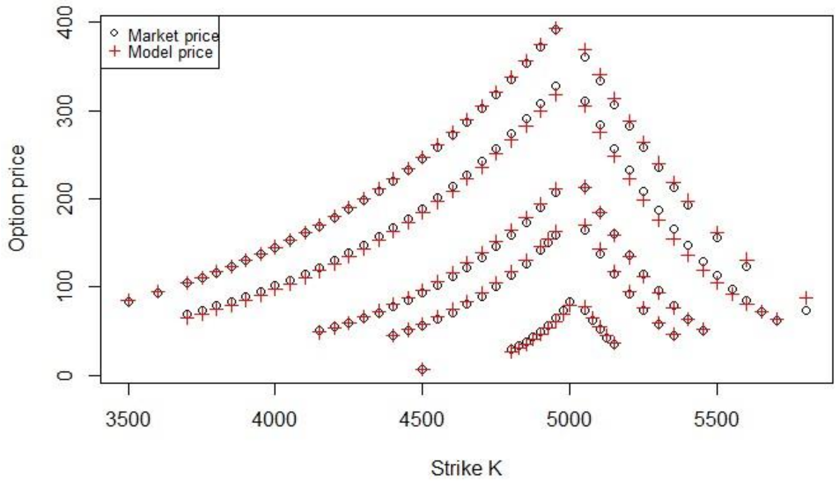
COMPARISON OF MODEL CALIBRATIONS (CAC 40, RMSE)

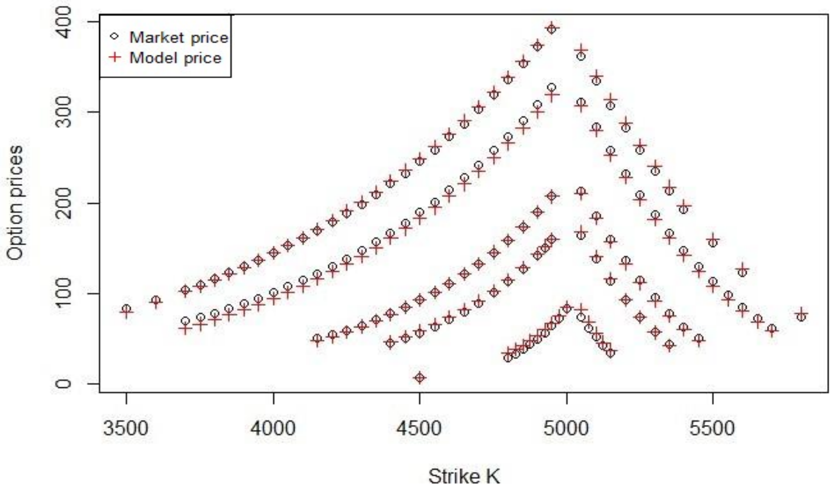
Heston : RMSE = 0.1193



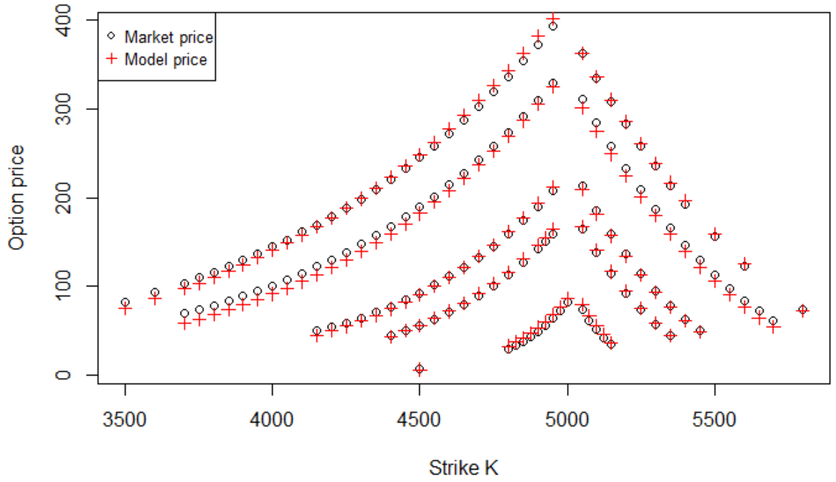
Bates : RMSE = 0.0863



Rough Heston : RMSE = 0.0929

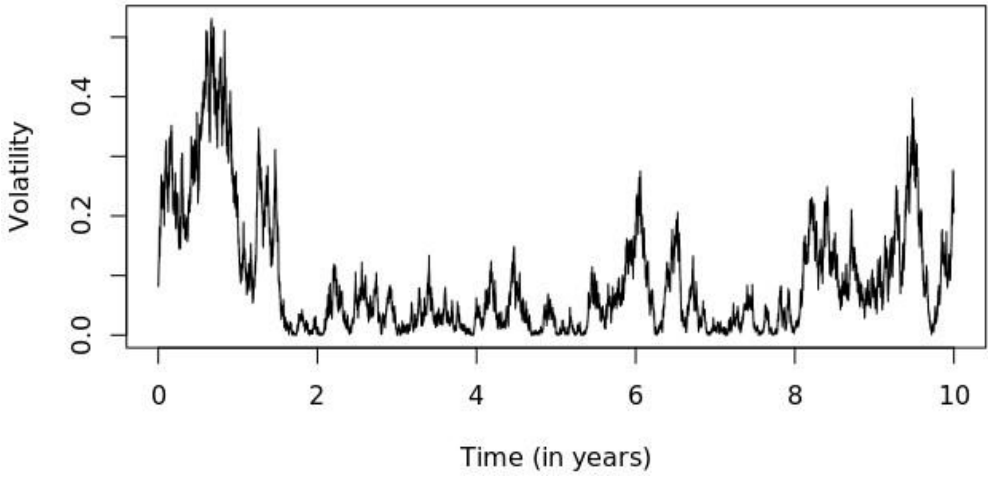


rBergomi : RMSE = 0.11315

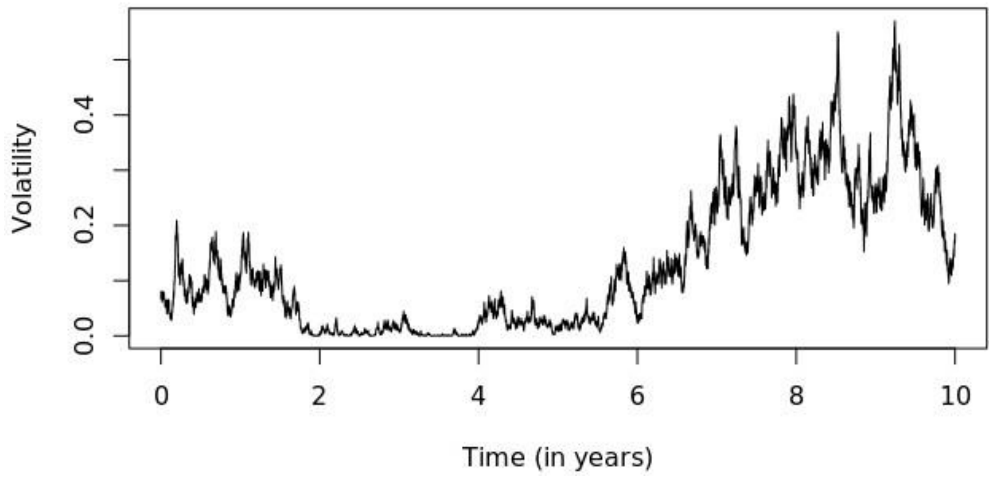


COMPARISON OF VOLATILITY SAMPLE PATHS

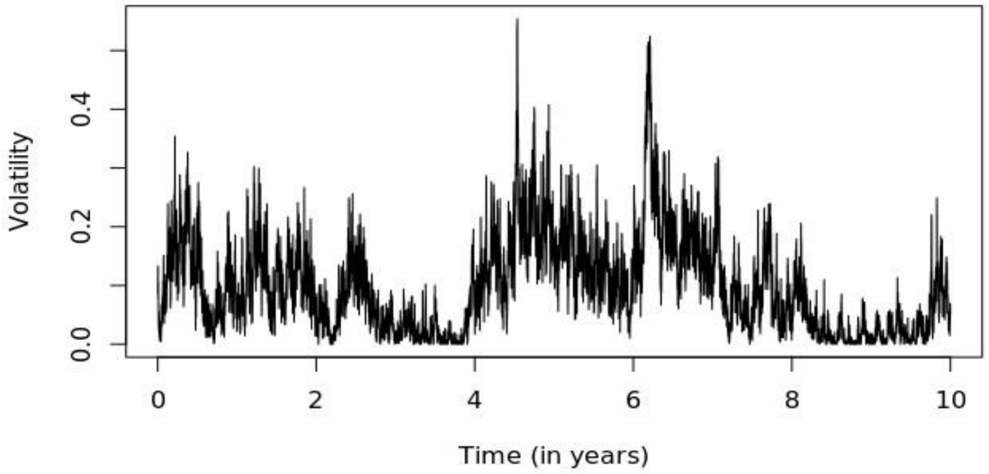
Heston : Milstein Scheme, *H=1/2*



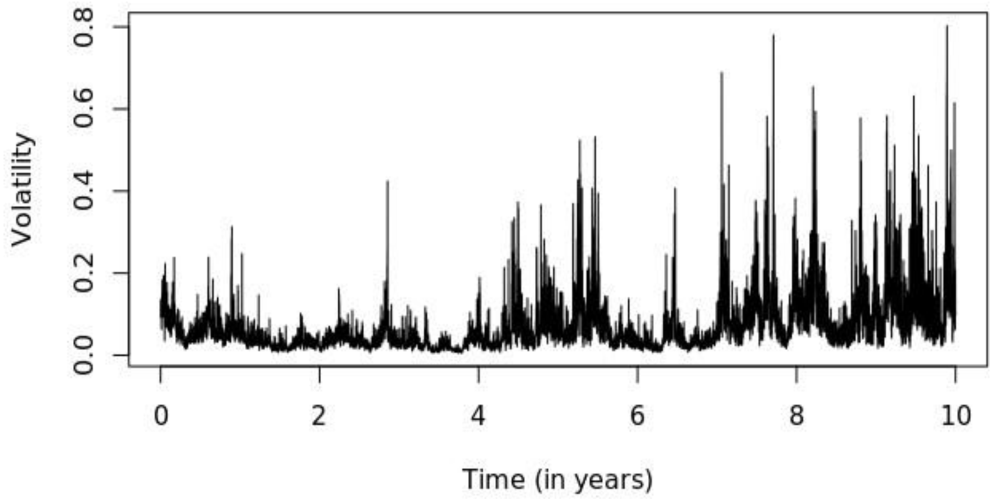
Bates : Milstein Scheme, *H = 1/2*



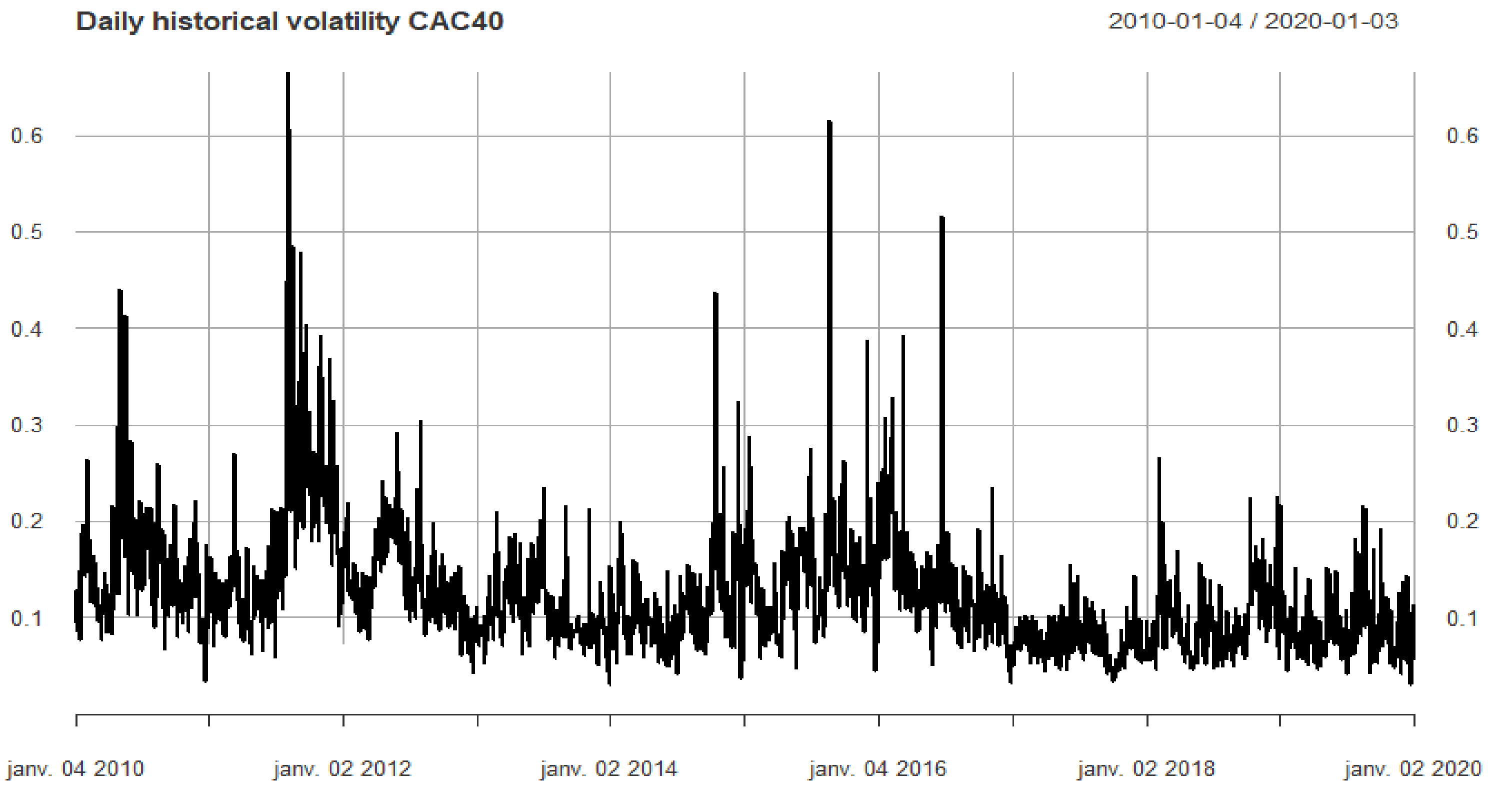
Rough Heston : Euler Scheme, *H=0.123*



rBergomi : Hybrid Scheme, *H=0.150*



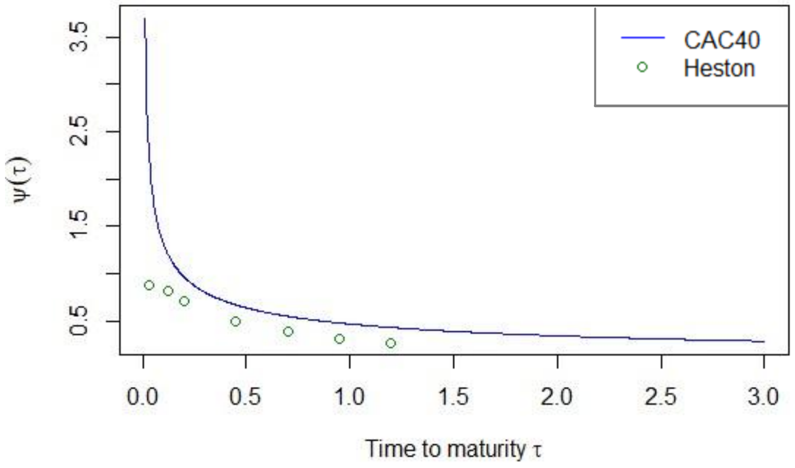
OBSERVED HISTORICAL VOLATILITY CAC40



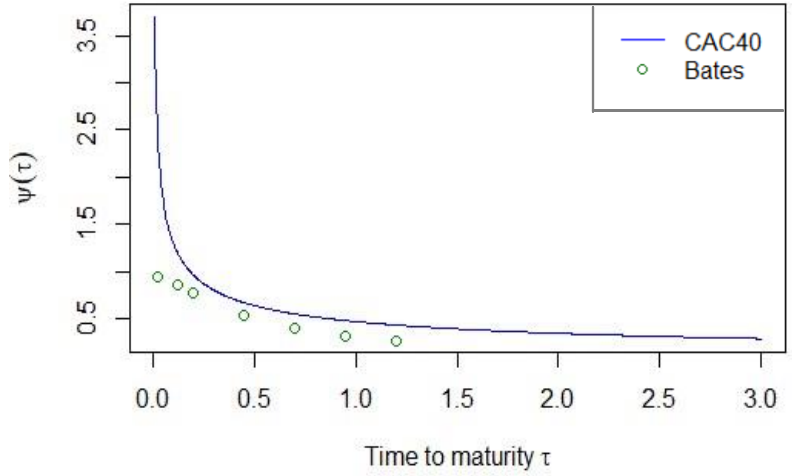
→ Visually, same volatility sample paths as rough models.

COMPARISON OF ATM VOLATILITY SKEWS

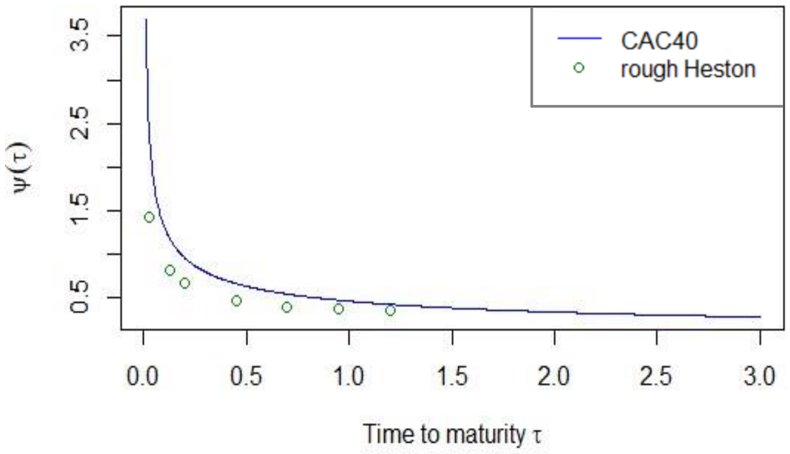
Heston : ATM volatility skews



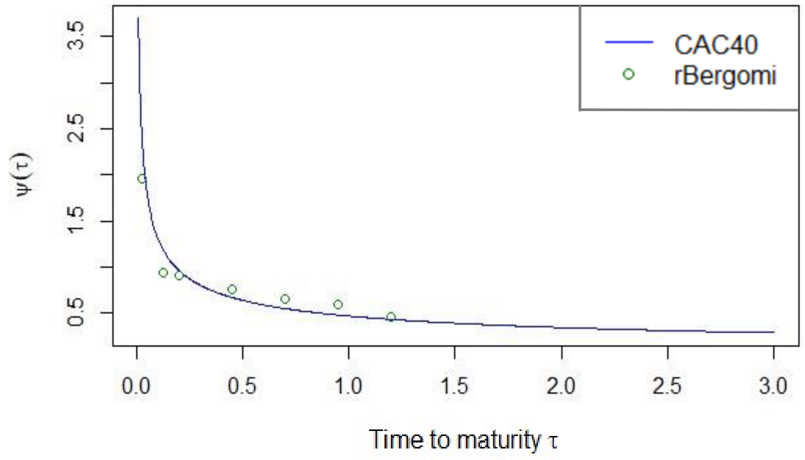
Bates : ATM volatility skews



Rough Heston : ATM volatility skews



rBergomi : ATM volatility skews



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EQUITY-LINKED LIFE INSURANCE CONTRACT

- **Endowment insurance** with maturity *T* where the benefits depend on the value of the fund *F(t).*

- Minimal annual return $\boldsymbol{\kappa_g}$ and maximal annual return $\boldsymbol{\kappa_m}$ on *F(t)* with participation rate $\boldsymbol{\eta}$.

- The survival benefit is given by $F_T^e \times \mathbb{I}\{t \geq T\}$ and the death benefit by $F_t^e \times \mathbb{I}\{t < T\}$ where :

$$F_t^e = F_0 \prod_{u=1}^{\lfloor t \rfloor} \min\left\{e^{\kappa_m} \; ; \; \max\left\{1 + \eta\left(\frac{S_u}{S_{u-1}} - 1\right) \; ; \; e^{\kappa_g}\right\}\right\}$$

- The faire value is given by discounting the expected benefits under a risk-neutral measure $\mathbb{P}^*$ with mortality modeled by a Poisson process (Makeham force of mortality μ_x).

FAIR VALUES OF LIFE INSURANCE CONTRACTS

- 50-year-old female policyholder, $F(0) = 10\ 000$ € , $\kappa_m = 20\%$ et $\eta = 80\%$.

- Fair value FV_0 for different maturities T with $\kappa_g = 1\%$:

$k_g = 1\%$	Heston	Bates	rBergomi	rough Heston
$T = 5$	14 036.57 €	13 457.49 €	12 291.17 €	13 529.65 €
$T = 10$	19 372.58 €	17 696.27 €	14 909.54 €	14 063.09 €
$T = 20$	35 631.76 €	29 613.34 €	20 639.81 €	17 823.38 €

Most market-consistent and accurate fair values !

→ Lower fair values of **rough-type models** compared with the Heston and Bates models.

→ Higher fair values of the **rBergomi** model compared with the **rough Heston** model for **large maturities *T*** (non-stationarity).

WHY USING THE ROUGH HESTON MODEL ?

- The <u>**rough Heston**</u> model allows :

- To better reproduce the **observed historical volatility.**

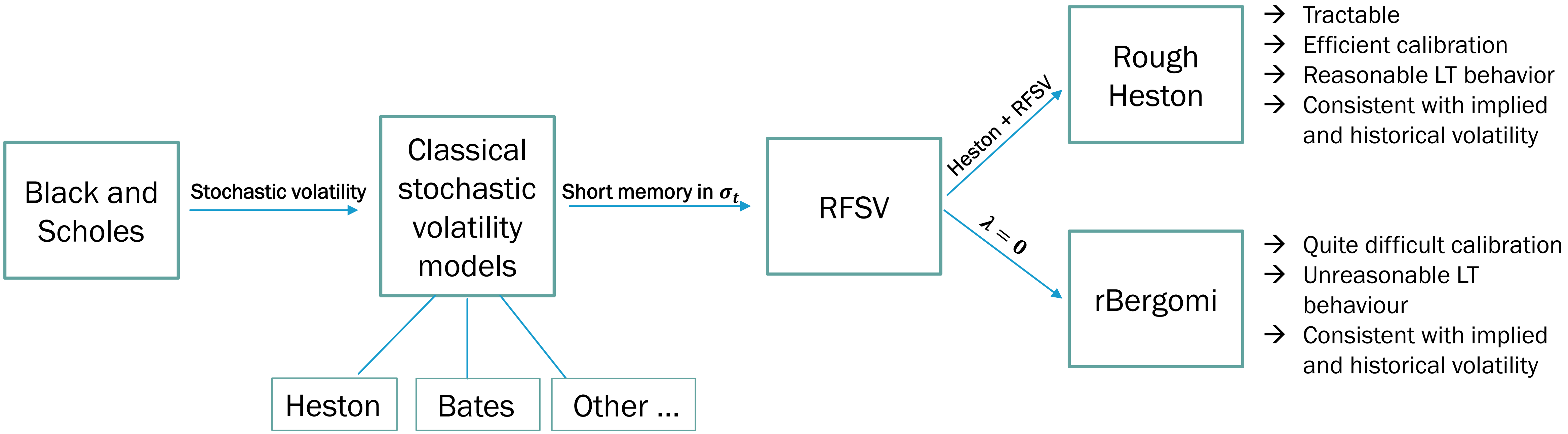
- A better modeling of the **implied volatility surface** (ATM volatility skews).

- An easy, **efficient and stable calibration** method with only 3 parameters.

- Reasonable long-term properties due to its **stationary volatility** process.

→ The <u>**rough Heston**</u> tends to outperform existing models in terms of **long-term pricing of insurance contrats**

TO SUMMARIZE …



Thank you for your attention !