Pricing of Derivatives with Memory

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Motivation

In the **commodity markets**, several findings emerge:

1. There are **storage costs**
2. There is **no dividend**
3. The market is no longer complete and then the risk-neutral probability measure is **no longer unique** $\rightarrow$ New probability measure.
4. Some markets are associated with **mean-reversion** features.
5. Some markets are driven by **long-range dependency** structures.

$$\Downarrow$$

Benth (2020) proposes to study a commodity price model to **combine** these different observations.

Lévy Process

Let introduce $(L_t)_{0\leq t\leq T}$ be a **Lévy process** with the characteristic triplet (μ, σ^2, ν) such that

$$L_t = \mu t + \sigma W_t + \int_0^t \int_{\mathbb{R}} z N(\mathrm{d}s \times \mathrm{d}z), \qquad 0 \leq t \leq T,$$

with an **asymmetric double exponential law** concerning the distribution of the size of the jumps

$$f_Y(y) = p\eta_1 e^{-\eta_1 y} \mathbb{1}_{\{y\geq 0\}} + (1-p)\eta_2 e^{\eta_2 y} \mathbb{1}_{\{y<0\}}.$$

Advantages :

1. Fat tails of the distribution
2. Asymmetric distribution
3. Implied volatility skew

Generalized Langevin Equation

Let introduce $(S_t)_{0\leq t\leq T}$ **the price in commodity market** and $X_t := \log(S_t)$ **the log-price**, such as $(X_t)_{0\leq t\leq T}$ is a generalized Langevin equation of the form

$$\mathrm{d}X_t = \beta(\theta - X_t)\mathrm{d}t + \left(\int_0^t M(t-u)X_u \mathrm{d}u\right)\mathrm{d}t + \chi(t-)\mathrm{d}L_t,$$

Takahashi (1996) gave the following solution for **the Laplace-Fourier transform** of X_t :

$$\mathcal{L}[X_t] = \left[X_0 + \frac{\theta \times \beta}{s} + \chi(t-)\mathcal{L}\left[\frac{\mathrm{d}L_t}{\mathrm{d}t}\right]\right]H(s),$$

where

$$H(s) = \frac{1}{s - \mathcal{L}[M(s)] + \beta}$$

which allows to get **an expression for** X_t :

$$X_t = X_0\mathcal{L}^{-1}[H(s)] + \beta\theta\int_0^t \mathcal{L}^{-1}[H(s)]_{t-u}\mathrm{d}u + \int_0^t \chi(u-)\mathcal{L}^{-1}[H(s)]_{t-u}\frac{\mathrm{d}L_u}{\mathrm{d}u}\mathrm{d}u,$$

First Kernel : Delta Function

The first option is a function that introduces an **instantaneous influence of the past evolution**, i.e. the Dirac Delta function at zero of the form $M(t) = a\delta_0(t)$ such as

$$\mathrm{d}X_t = \beta(\theta - X_t)\mathrm{d}t + \left(\int_0^t a\delta_0(t-u)X_u \mathrm{d}u \right)\mathrm{d}t + \chi(t-)\mathrm{d}L_t.$$

Using the Laplace-Fourier table, the process X_t becomes :

$$X_t = e^{-(\beta-a)t}X_0 + \beta\theta\left[\frac{1-e^{-(\beta-a)t}}{\beta-a}\right] + \int_0^t \chi(u-)e^{-(\beta-a)(t-u)}\mathrm{d}L_u$$

Similarities with the Ornstein-Ulhenbeck model.

Second Kernel : Negative Exponential

The second option is a function that introduces a **persistent influence over a relatively short period of time**. If $M(t) = ae^{-bt}$ with $b > 0$ and $b^2 - 4a > 0$, we get :

$$\mathrm{d}X_t = \beta(\theta - X_t)\mathrm{d}t + \left(\int_0^t ae^{-b(t-u)} X_u \mathrm{d}u \right)\mathrm{d}t + \chi(t-)\mathrm{d}L_t.$$

Using the Laplace-Fourier table, the expression of the process X_t becomes :

$$X_t = \frac{1}{s_1 - s_2} \times \Bigg(X_0 \Big[(s_1 + b)e^{s_1 t} - (s_2 + b)e^{s_2 t} \Big] + \beta\theta \Bigg\{ \frac{(s_1 + b)(e^{s_1 t} - 1)}{s_1} - \frac{(s_2 + b)(e^{s_2 t} - 1)}{s_2} \Bigg\} + \int_0^t \chi(u-) \Big\{ (s_1 + b)e^{s_1 (t-u)} - (s_2 + b)e^{s_2 (t-u)} \Big\} \mathrm{d}L_u \Bigg)$$

where s_1, s_2 are the two roots of the denominator in the inverse Laplace-Fourier transform... **not very tractable**.

Third Kernel : Negative Power

A negative power function is used when the influence of the past **persists over a longer period of time**. If $M(t) = at^{-\alpha}$ with the condition $0 < \alpha < 1$:

$$\mathrm{d}X_t = \beta(\theta - X_t)\mathrm{d}t + \left(\int_0^t a(t-u)^{-\alpha} X_u \mathrm{d}u\right)\mathrm{d}t + \chi(t-)\mathrm{d}L_t,$$

By using the same procedure as for the previous expression with $\mathcal{L}[M(s)] = \frac{a\Gamma(1-\alpha)}{s^{1-\alpha}}$ it follows :

$$\begin{aligned}\mathcal{L}^{-1}[H(s)] &= \mathcal{L}^{-1}\left[\frac{1}{s - \mathcal{L}[M(s)] + \beta}\right] \\ &= \mathcal{L}^{-1}\left[\frac{s^{1-\alpha}}{s^{2-\alpha} + \beta s^{1-\alpha} - a\Gamma(1-\alpha)}\right]\end{aligned}$$

There is **no closed formula** for this inverse Laplace-Fourier transform. Different alternatives have been suggested but... **not tractable**.

Power Kernel (cont.)

Case 1 : Benth & al (2020)

Approximation of this Laplace inverse function:

$$\mathcal{L}^{-1}[H(s)] = \sum_{i=0}^{\infty} w_i(\alpha)(at^{2-\alpha})^i,$$

where $w_i(\alpha)$ represent the weights and have the following recursive relations :

$$w_i(\alpha) = w_{i-1}(\alpha) \frac{\Gamma\Big(1+(i-1)(2-\alpha)\Big)}{\Gamma\Big(1+i(2-\alpha)\Big)} \Gamma(1-\alpha),$$

with $w_0(\alpha) = 1$.

Case 2 : Hainaut (2021)

Introduction of a new kernel function:

$$M(t) = \frac{at^{\alpha-2}}{\Gamma(\alpha-1)},$$

for any $1 < \alpha < 2$. Leading to:

$$\mathcal{L}^{-1}[H(s)] = \mathcal{E}_\alpha[-at^\alpha]$$

where $\mathcal{E}(.)$ is the Mittag-Feller function, such as :

$$\mathcal{E}_\alpha[-at^\alpha] = \sum_{n=0}^{\infty} \frac{(-at^\alpha)^n}{\Gamma(n\alpha+1)}.$$

Case 3 : Takahashi (1996)

Introduction of a specific kernel function:

$$M(t) = at^{-0.5},$$

but leading to the necessary of using:

$$\mathcal{L}^{-1}\Big[\frac{1}{\sqrt{s}-a}\Big] = \frac{1}{\sqrt{\pi t}} + ae^{a^2 t}\operatorname{erfc}(-a\sqrt{t}).$$

$\Longrightarrow$ **Loss of generality, not tractable form and difficult to interpret.**

Pathwise Comparison

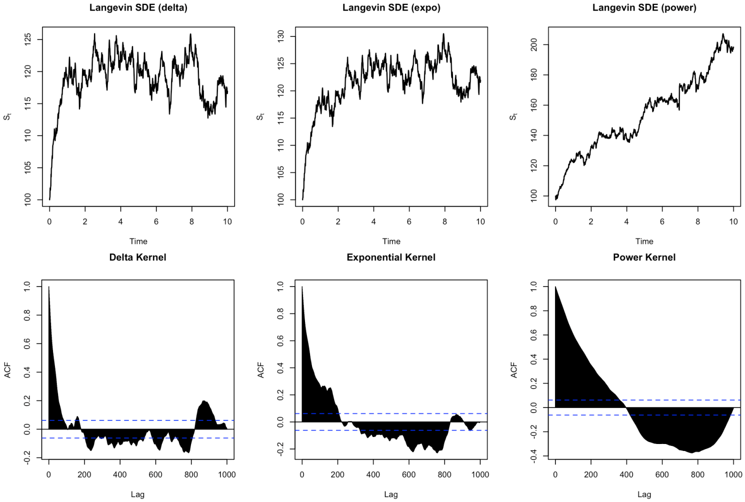

Figure: A simulation of the price of a commodity according to the three different kernels. **The starting level is set lower than the mean reversion level.**

Change of Measure

By defining the numeraire of this pricing measure such that

$$B_t := \exp\left\{\int_0^t r_s \mathrm{d}s\right\}$$

and the **convenience yield**

$$\rho_t = \beta(\theta - X_t) + \int_0^t M(t-u) X_u \mathrm{d}u.$$

The dynamics of the **discounted commodity price** under the pricing measure $\mathbb{Q}$ becomes

$$d\tilde{S}_t = \tilde{S}_t \left\{ \rho_t \mathrm{d}t + \sigma \mathrm{d}W_t^{\mathbb{Q}} + \int_{\mathbb{R}} (e^z - 1)\tilde{N}^{\mathbb{Q}}(\mathrm{d}t, \mathrm{d}z) \right\},$$

and the **commodity price**

$$\mathrm{d}S_t = S_t \left\{ \left(r_t + \beta(\theta - X_t) + \int_0^t M(t-u) X_u \mathrm{d}u \right) \mathrm{d}t + \sigma \mathrm{d}W_t^{\mathbb{Q}} + \int_{\mathbb{R}} (e^z - 1)\tilde{N}^{\mathbb{Q}}(\mathrm{d}t, \mathrm{d}z) \right\},$$

which has a return equal to the **sum of the interest rate and of the convenience yield** $\implies$ which is assimilated to **memory**.

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Data

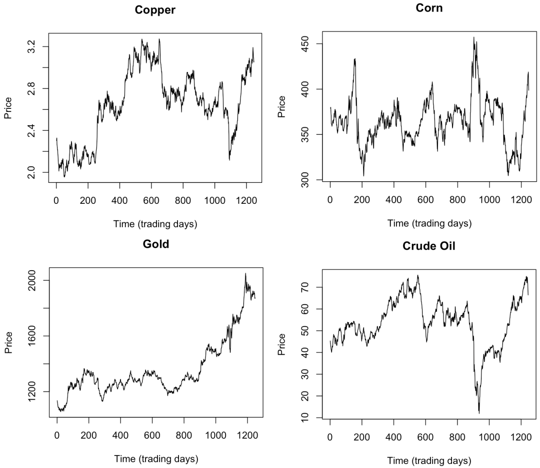

Figure: **Prices of the different commodities markets studied between November 2015 and November 2020.** They are all reported in USD except for Corn, which is reported in US cents.

Optimal Models Selected

Commodity Market	**Gold (Exponential)**	**Oil (Delta)**	**Corn (Kou)**	**Copper (Delta)**
a	0.3748	0.1467	/	1.1447
b	2.335	/	/	/
θ	0.2472	0.1644	/	0.0905
β	1.0507	2.3533	/	1.8553
λ	29.681	36.642	159.194	49.907
p	0.7331	0.227	0.489	0.473
η_1	128.294	15.203	113.543	117.233
η_2	148.617	51.687	118.509	130.356
σ	0.1212	0.224	0.106	0.149
Memory	**Persistent**	**Instantaneous**	**No memory**	**Instantaneous**

Table: Parameters obtained with respect to the optimal models for each market.

Implied Volatility

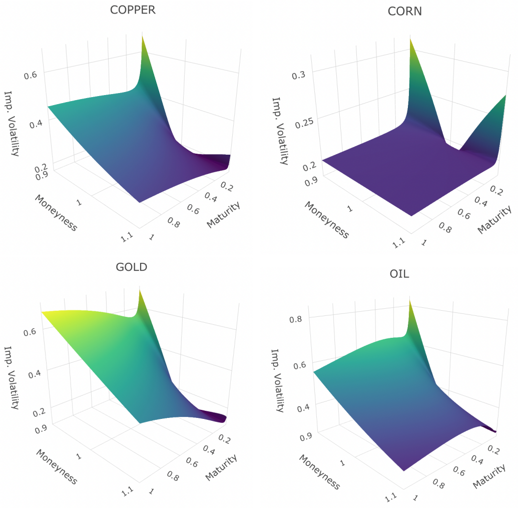

Figure: **Implied volatility surface** of each commodity.

- **CORN (KOU):**
  No mean-reversion ⟶ Smile for small maturities
- **COPPER / GOLD / OIL:**
  Skewed jump distributions ⟶ Skew
- **COPPER / GOLD / OIL:**
  Mean-reversion ⟶ Convex volatilities over time
- **COPPER / GOLD / OIL:**
  Mean-reversion ⟶ Volatility inversely proportional to moneyness

Vanilla Option Pricing

Model	Gold	Corn	Oil	Copper
Kou	10.945	9.185	12.223	10.284
Delta Kernel	11.178	8.983	12.077	10.483
Expo Kernel	11.461	9.918	11.310	9.401

Table: **Call option prices** computed with the optimal parameters derived earlier for each of the markets and each of the models for a maturity of one month.

Vanilla options only focused on **marginal distributions at maturity**

$\Downarrow$

Forget the « **path-dependent** » effects that characterise memory processes

Exotic Pricing

Motivation to price a **path-dependent derivative** relevant for commodities markets

$$\Downarrow$$

Barrier Reverse Convertible with the Fair Value at inception:

$$FV_{t=0} = \underbrace{C \sum_{i=1}^{n} \left(e^{-r(t_i)\ t_i} \right)}_{n \text{ coupons}} + \underbrace{\frac{N}{S_0} e^{-r(T)T} \mathbb{E}^{\mathbb{Q}} \left[S_0 - \mathbb{1}_{\{L \leq H\}} \times \left(S_0 - S_T \right)_+ \right]}_{\text{Repayment of the principal (or a part)}}$$

Three different cases are possible for the repayment depending on the evolution of $(S_t)_{0 \leq t \leq T}$:

$$\text{Payoff}^{BRC} = \underbrace{N \times \mathbb{1}_{\{L > H\}}}_{\text{Never crosses}} + \underbrace{N \times \mathbb{1}_{\{L \leq H\}} \times \mathbb{1}_{\{S_T > S_0\}}}_{\text{Crosses but ends up above}} + \underbrace{\frac{N \times S_T}{S_0} \times \mathbb{1}_{\{L \leq H\}} \times \mathbb{1}_{\{S_0 \geq S_T\}}}_{\text{Crosses and ends up below}}$$

with H is a lower barrier and L the minimum value of S_t $\forall 0 \leq t \leq T$.

Exotic Pricing (cont.)

Barrier Reverse Convertible (Down-and-In Barrier Put Option, i.e. DIBP):

$$FV_{t=0} = C \sum_{i=1}^{n} \left(e^{-r(t_i)\ t_i} \right) + \frac{N}{S_0} e^{-r(T)T} \mathbb{E}^{\mathbb{Q}} \left[S_0 - \mathbb{1}_{\{L \leq H\}} \times \left(S_0 - S_T \right)_+ \right]$$

The **value of the fair coupons** is therefore such as (for $N = 1$):

$$C = \frac{(1-x) - e^{-r(T)T} + \frac{1}{S_0}\text{DIBP}_{t=0}}{\sum_{i=1}^{n} \left(e^{-r(t_i)\ t_i} \right)}, \qquad \text{with the banking fees } x.$$

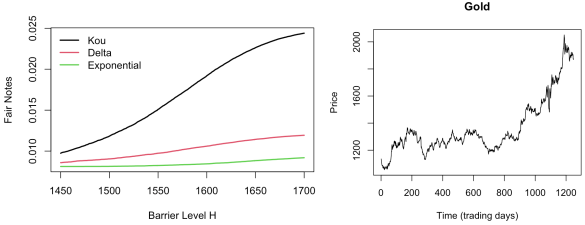

Figure: Comparison of coupon rates for the **Gold market**.

$\Longrightarrow$ **More market-consistent and have more accurate fair values.**

Conclusion

Various points should be highlighted:

- Non unique risk-neutral measure → **Pricing probability measure**.
- **Convenience Yield**: no dividend and costs of storage etc.
- Different kinds of **time dependency**: short (Oil) or long (Gold) term structures → better reproduce the observed historical properties.
- The interest of using non-Markovian models in the pricing of derivatives (path-dependent) → **market-consistency**.

Future tracks:

- Model implementation for the **convenience yield** and the **interest rate**.
- Focus on the very important topic of **hedging**.

Thank you for your attention !

Delta Kernel (cont.)

The characteristic function of the process, considering that we are at inception :

$$\Phi_{X_t}(u) = \exp\left(iu\left(e^{-(\beta-a)t}X_0 + \left[\beta\theta + \mu\right]\frac{1-e^{-(\beta-a)t}}{\beta - a}\right) - \frac{\sigma^2 u^2 (1-e^{-2(\beta-a)t})}{4(\beta-a)} \right.$$
$$\left. + \lambda\left\{ \frac{p}{\beta-a}\log\left(\frac{\eta_1 - iue^{-(\beta-a)t}}{\eta_1 - iu}\right) + \frac{(1-p)}{\beta-a}\log\left(\frac{\eta_2 + iue^{-(\beta-a)t}}{\eta_2 + iu}\right)\right\}\right).$$

For ease of calibration in the practical part, we assume here that the mean of the diffusion term is zero, such as

$$\begin{aligned}\mathbb{C}(X_t X_s) &= \mathbb{E}\left[\int_0^t e^{-(\beta-a)(t-u)}\mathrm{d}L_u \int_0^s e^{-(\beta-a)(s-u)}\mathrm{d}L_u\right] \\ &= \left(\sigma^2 + \lambda\mathbb{E}[Y^2]\right) \times \frac{e^{-(\beta-a)(t-s)} - e^{-(\beta-a)(t+s)}}{2(\beta-a)}\end{aligned}$$

and finally the autocorrelation

$$\rho_k = \frac{e^{-(\beta-a)k} - e^{-(\beta-a)(2t-k)}}{\sqrt{1 - e^{-2(\beta-a)t} - e^{-2(\beta-a)(t-k)} + e^{-2(\beta-a)(2t-k)}}}.$$

Exponential Kernel (cont.)

The characteristic function of the process, considering that we are at inception :

$$\Phi_{X_t}(u) = \exp\Bigg(\frac{iu}{s_1 - s_2} \times \Bigg[X_0[(s_1+b)e^{s_1 t} - (s_2+b)e^{s_2 t}] + (\beta\theta + \mu) \times \Big\{\frac{(s_1+b)(e^{s_1 t}-1)}{s_1} - \frac{(s_2+b)(e^{s_2 t} -}{s_2}$$

$$- \frac{u^2\sigma^2}{4(s_1-s_2)^2}\Big(\frac{(s_1+b)^2(e^{2s_1 t}-1)}{s_1} - \frac{(s_2+b)^2(e^{2s_2 t}-1)}{s_2}\Big)$$

$$+ \lambda\Big\{\frac{p}{s_1} \times \log\Big(\frac{\eta_1(s_1-s_2) - iu(s_1+b)}{\eta_1(s_1-s_2) - iu(s_1+b)e^{s_1 t}}\Big) + \frac{(1-p)}{s_1} \times \log\Big(\frac{\eta_2(s_1-s_2) + iu(s_1+b)}{\eta_2(s_1-s_2) + iu(s_1+b)e^{s_1 t}}\Big)$$

$$- \frac{p}{s_2} \times \log\Big(\frac{\eta_1(s_1-s_2) - iu(s_2+b)}{\eta_1(s_1-s_2) - iu(s_2+b)e^{s_2 t}}\Big) - \frac{(1-p)}{s_2} \times \log\Big(\frac{\eta_2(s_1-s_2) + iu(s_2+b)}{\eta_2(s_1-s_2) + iu(s_2+b)e^{s_2 t}}\Big)\Big\}\Bigg)$$

Assuming that the mean of the diffusion term is zero, such as

$$\mathbb{C}(X_t X_s) = \frac{\Big(\sigma^2 + \lambda\mathbb{E}[Y^2]\Big)}{(s_1-s_2)^2} \times \Big[\frac{(s_1+b)^2}{2s_1}(e^{s_1(t+s)} - e^{s_1(t-s)}) - \frac{(s_1+b)(s_2+b)}{(s_1+s_2)}(e^{s_1 t + s_2 s} - e^{s_1(t-s)})$$

$$- \frac{(s_1+b)(s_2+b)}{(s_1+s_2)}(e^{s_2 t + s_1 s} - e^{s_2(t-s)}) + \frac{(s_2+b)^2}{2s_2}(e^{s_2(t+s)} - e^{s_2(t-s)})\Big]$$

Exponential Kernel (cont.)

And finally the autocorrelation

$$\rho_k = \frac{\frac{(s_1+b)^2}{2s_1}(e^{s_1(2t-k)} - e^{s_1 k}) - \frac{(s_1+b)(s_2+b)}{(s_1+s_2)}\left((e^{s_1 t+s_2(t-k)} - e^{s_1 k}) + (e^{s_2 t+s_1(t-k)} - e^{s_2 k})\right) + \frac{(s_2+b)^2}{2s_2}(e^{s_2(2t-k)} - e^{s_2 k})}{\sqrt{\varphi(t)} \times \sqrt{\varphi(t-k)}},$$

with

$$\varphi(t) = \left((s_1+b)^2\left[\frac{e^{2s_1 t}-1}{2s_1}\right] - 2(s_1+b)(s_2+b)\left[\frac{e^{(s_1+s_2)t}-1}{s_1+s_2}\right] + (s_2+b)^2\left[\frac{e^{2s_2 t}-1}{2s_2}\right]\right).$$

Calibration

1. Define the log-price:
$$X_j = \log(P_j/P_0)$$
where P_j is the j-*th* price.

2. Obtain the parameters of the autocorrelation by RMSE:
$$RMSE = \sqrt{\sum_{j=1}^{N} \frac{1}{N}(\rho_j^{market} - \hat{\rho}_j^{model})^2}.$$
Then we obtain the parameters such as : $\Theta_1^* = \arg\min_{\Theta_1} \mathcal{L}(\rho_j^{market}, \hat{\rho}_j^{model}(\Theta_1))$.

3. De-mean the processes : for the Delta kernel as
$$Y_j = X_j - \left(\frac{\beta^*\theta}{\beta^* - a^*}\right)\left(1 - e^{-(\beta^* - a^*)t_j}\right),$$
and for the Exponential kernel as,
$$Y_j = X_j - \frac{\beta^*\theta}{s_1^* - s_2^*}\left\{\frac{(s_1^* + b^*)(e^{s_1^* t_j} - 1)}{s_1^*} - \frac{(s_2^* + b^*)(e^{s_2^* t_j} - 1)}{s_2^*}\right\}.$$
Then we get $\Theta_2^* = (\theta^*)$.

Calibration (cont.)

4. Approximate the process Z_j for the Delta kernel as

$$
\begin{aligned}
Z_j &= Y_j - Y_{j-1}e^{-(\beta^*-a^*)\Delta_t} \\
&= \int_0^{t+\Delta t} e^{-(\beta^*-a^*)(t+\Delta_t-u)}\mathrm{d}L_u - \int_0^t e^{-(\beta^*-a^*)(t-u)}\mathrm{d}L_u \\
&\approx \int_0^{\Delta_t} e^{-(\beta^*-a^*)(\Delta_t-u)}\mathrm{d}L_u
\end{aligned}
$$

and for the Exponential kernel as,

$$
\begin{aligned}
Z_j &= Y_j - Y_{j-1}\left(\frac{1}{s_1^*-s_2^*}\times\left[(s_1^*+b^*)e^{s_1^*\Delta_t}-(s_2^*+b^*)e^{s_2^*\Delta_t}\right]\right) \\
&= \frac{1}{s_1^*-s_2^*}\times\left\{\int_0^{t+\Delta_t}\left((s_1^*+b^*)e^{s_1^*(t+\Delta_t-u)}-(s_2^*+b^*)e^{s_2^*(t+\Delta_t-u)}\right)\mathrm{d}L_u\right. \\
&\qquad \left. - \int_0^t\left((s_1^*+b^*)e^{s_1^*(t-u)}-(s_2^*+b^*)e^{s_2^*(t-u)}\right)\mathrm{d}L_u\right\} \\
&\approx \frac{1}{s_1^*-s_2^*}\times\int_0^{\Delta_t}\left((s_1^*+b^*)e^{s_1^*(\Delta_t-u)}-(s_2^*+b^*)e^{s_2^*(\Delta_t-u)}\right)\mathrm{d}L_u.
\end{aligned}
$$

Calibration(cont.)

5. Maximise the log-likelihood:

$$\Theta_3^* = \arg\max_{\Theta_3} \ \log \hat{f}(\Theta_3; \mathbf{Z}), \tag{7.1}$$

by numerically inverting the characteristic function.

Optimal Fitted Densities (Noise : Lévy process)

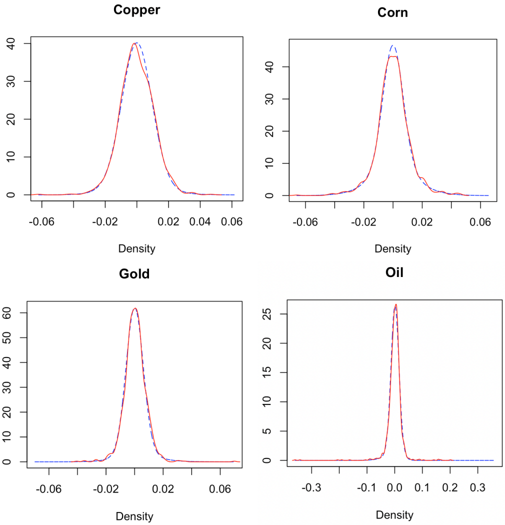

Figure: Empirical densities and those approximated by the optimal model for each market.

Moments of the different market

	Commodity Market	$\mathbb{E}[\Delta X_t]$	$\sqrt{\mathbb{V}[\Delta X_t]}$	$\mathbb{S}[\Delta X_t]$	$\mathbb{K}[\Delta X_t]$
Empirical	Gold (EXPO)	0.0003	0.0075	0.1809	7.778
	Copper (DELTA)	0.0001	0.0107	-0.0017	4.911
	Corn (KOU)	0	0.0112	-0.0476	5.363
	Oil (DELTA)	0.0002	0.0052	-0.0838	3.964
Model	Gold (EXPO)	0.0003	0.0075	0.2571	6.973
	Copper (DELTA)	0	0.0105	0	4.707
	Corn (KOU)	0	0.0110	-0.0537	5.215
	Oil (DELTA)	0	0.0048	0	3.871

Table: Moments of the daily log-returns of the markets and the fitted optimal models.

Simulations

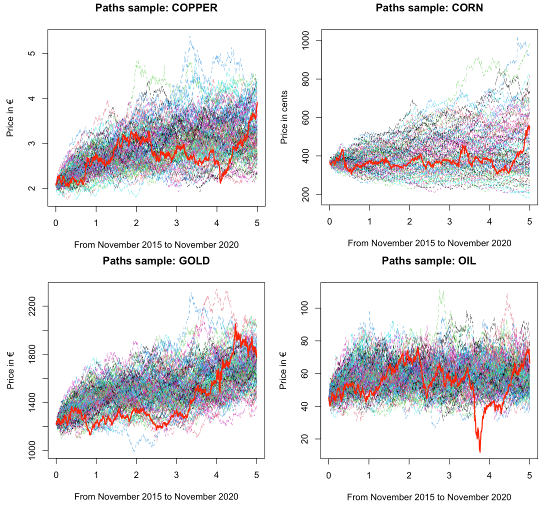

Figure: Simulation of 100 paths sample of the price of each commodity over the 5 years of observation and comparison with the price actually observed on the market.