Master thesis presentation

The COS method as a tool for pricing and hedging financial products within the framework of the Levy processes and development of the SIN method, an alternative method

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Plan

- Context
  - What is the general theme of this master thesis? What is the framework in which it takes place?
  - What can this master thesis contribute to the topic?
- Positioning of the problem
  - What is the current state of the literature on the subject?
  - What is the current state of knowledge?
- Methodology
  - Presentation of the main analyses conducted and the results obtained
- SIN Method
  - Introduction of a new method for retrieving density functions in Levy processes framework
  - New approach to valuation of GMWBs
- Conclusions

Context

- GMxB contracts are unit-linked policies to which **various minimum guarantees are added**.
- The most popular of these are GMWBs, as **they guarantee periodic withdrawals of a minimum amount, regardless of the performance of the underlying fund.**
- These contracts have become very popular in the USA over the last few decades, for several reasons :
  - They allow policyholders to **benefit from investing in the financial markets, while protecting against potential poor returns** as well as the current high volatility.
  - They offer flexibility by allowing the policyholder to choose the fund in which he invests.
  - They enable policyholders **to build up additional retirement savings** and thus protect themselves against longevity risk.
- In most European countries (including Belgium), demographics indicate that **people are living longer and longer**, and **traditional pension plans are facing increasing difficulties**.

- In this context, many people are turning to insurers to build up additional savings, and **these new products are emerging as a viable alternative for retirees**.
- However, **the pricing of these products can be complicated**. This is why **the majority of the literature assumes a classical BS model** to represent fund dynamics.
- **Levy processes** are processes that **more faithfully reflect the empirical behaviour of financial assets**, but which **make pricing much more complicated, requiring the use of** a mathematical tool derived from physics and called **the Fourier transform**.
- Nevertheless, in 2008, F. Fang & C.W. Osterlee proposed **a new alternative method** to the Fourier transform, called the **COS method**.

Contribution of this master thesis:

- Use of the COS method through these different applications in order **to test its reliability and efficiency**, for example option pricing and hedging.
- **Comparison of the COS method and the Fourier transform** to approximate the density function of different Levy processes
- **Introduction of a new method, called the SIN method**, and **comparison of the results obtained** with the COS method and the Fourier transform, **in particular for GMWB pricing**.
- **The main conclusion is that these two methods offer superior results to the classical Fourier transform method**, especially for the pricing of GMWB contracts.

Positioning of the problem

- Limitations of the Black-Scholes model :
  - Gaussian returns
  - Continuous path
  - Constant volatility

- Levy processes are entirely characterized by their characteristic triplet (σ, γ, ν) thanks to the *Lévy-Khintchine representation* :

$$\Phi_t(z) = E[e^{iZX_t}] = e^{\Psi(z)t}$$

$$\Psi(z) = -\frac{1}{2}\sigma^2 z^2 + i\gamma z + \int_{\mathbb{R}} (e^{izx} - 1 - izx\mathbb{1}_{|x|<1})\nu(dx)$$

- The characteristic function is linked to the density function f_t by the *Fourier transform*

$$\Phi_t(\omega) = \int_{\mathbb{R}} f_t(x) e^{i\omega x} dx$$

- Traditionally, the density function is found by calculating the inverse Fourier transform of the characteristic function.

$$f_t(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-i\omega x} \Phi_t(\omega) d\omega$$

- F.Fang and C.W. Osterlee (2008) have developed a method to find the density function f of a random variable X of characteristic function Φ via a cosine expansion of the function :

$$\mathrm{f(x)} \approx \frac{2}{b-a} \sum_{n=0}^{N-1}{}' Re \left\{ \Phi \left(\frac{n\pi}{b-a} \right) exp \left(-i \frac{na\pi}{b-a} \right) \right\} cos \left(n\pi \frac{x-a}{b-a} \right)$$

- This method was used by J. Alsonso-Garcia, O. Wood and J. Ziveyi (2017) to price GMWB variable annuity contracts when the underlying follows a Levy process.

3 types of processes were considered :

1. The geometric Brownian movement (GBM) :

$$\Phi_t(u) = exp\left(iu(r+d)t - \tfrac{1}{2}u^2\sigma^2 t\right)$$

2. The Variance-Gamma process:

$$\Phi_t(u) = exp\left(iu(r+d)t\right) \times \left(1 - iu\mu\nu + \tfrac{1}{2}\sigma^2 u^2 \nu\right)^{\frac{-t}{\nu}}$$

3. The CGMY process:

$$\Phi_t(u) = exp\left(iu(r+d)t - \frac{1}{2}u^2\sigma^2 t\right) \times \\ exp\left(ct\Gamma(-Y)\left[(M-iu)^Y - M^Y + (G+1)^Y - G^Y\right]\right)$$

- The parameters used for each of these processes are as follows:

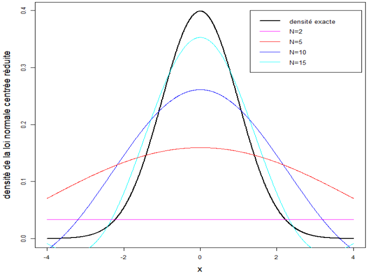
GBM	VG	CGMY
$\sigma = 0.1361$	$\sigma = 0.1301$ $\theta = -0.3150$ $\mu = 0.1753$	C=0.6817 G=18.0293 M=57.6250 Y=0.8000

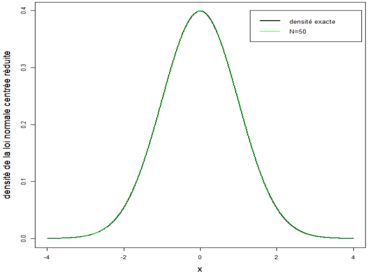
Parameters calibrated by Bacinello et al. (2014) on the S&P 500 and used by Alonso-Garcia et al. (2017) for the valuation of GMWB contracts using the COS method.

- The objective is to implement the classical Fourier transform method and the COS method to find the density function of these different processes, and to test the utility of the COS method compared to the classical Fourier transform (convergence speed, computation time,...)

Approximation of density function of common distributions

- The standard normal distribution



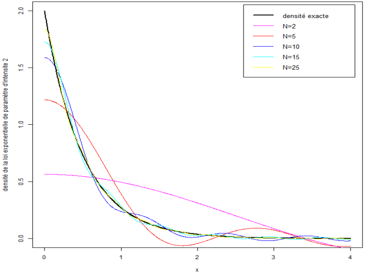


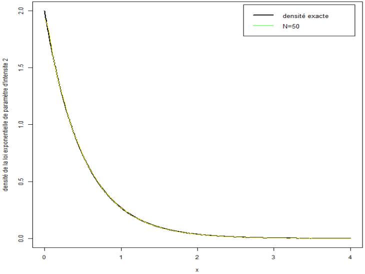
	N=5	N=25	N=50
max.errors (abs.value)	0.23932	0.00348	1.092×10^{-7}
comput. time (sec.)	0.37552	0.41553	0.46792

$\Rightarrow$ The COS method is efficient to quickly and accurately approximate the density of the normal distribution.

Approximation of density function of common distributions

- The exponential distribution ($\lambda = 2$)





	N=5	N=25	N=50
max.errors (abs.value)	1.43662	0.16448	0.08176
comput. time (sec.)	0.45662	0.51908	0.60774

$\Rightarrow$ The COS method is efficient to quickly and accurately approximate the density of the exponential distribution.

Valuation of options

- Price of a European call

	N=2	N=15	N=50	N=100
Option value	14.98964	3.10188	3.10369	3.10369
comput. time (sec.)	0.93362	0.93804	0.95169	0.96636

Approximation of the price of a European call of parameters $S_0 = 100$, $r = 2\%$, $T = 0.1$, $\sigma = 0.15$ and $K = 100$ by the COS method as a function of the parameter N (Black-Scholes value = 3.02757)

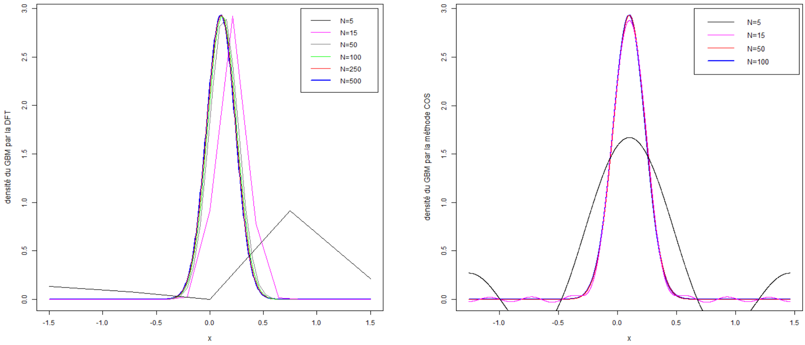
3 remarks:

- Very fast convergence : From $N = 2$ to $N = 15$, the price varies by 79.31%. From $N = 50$, the price stabilizes.
- Difference of 2.51% from Black-Scholes value.
- For a put of same parameters, difference of 0.46%

$\Rightarrow$ The COS method makes it possible to price options reliably.

Comparison of Fourier transform / Cos method for Levy process

- The geometric Brownian movement (GBM)



Approximation of the GBM density function by the DFT (left) and the COS method (right).

Comparison with the exact density :

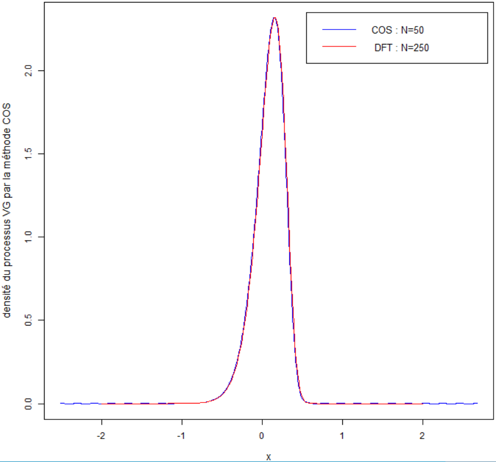
		N=15	N=50	N=100	N=500
max. errors	DFT	1.31800	0.39631	0.19576	0.03927
	COS	0.05272	3.33×10^{-14}	3.52×10^{-15}	3.52×10^{-15}

Several remarks:

1. For N fixed, the COS method is much more accurate.
2. Faster convergence of the COS method

Comparison of Fourier transform / Cos method for Levy process

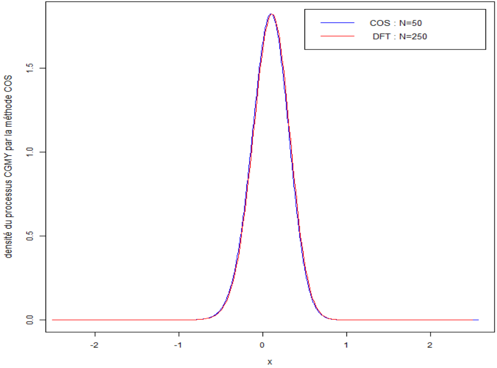
- The Variance-Gamma process (VG)



- No comparison with the exact density possible.
- Looks similar to the DFT for $N = 50$ instead of $N = 250$.
- A priori, the COS method with $N = 50$ approximates the density better than the DFT with $N = 250$.

Comparison of Fourier transform / Cos method for Levy process

- The CGMY process



- Similar conclusions to the VG process

$\Rightarrow$ The COS method is more efficient in terms of speed of convergence and accuracy of results to approximate the density of Levy processes compared to classical Fourier transform methods.

Introduction to the SIN method

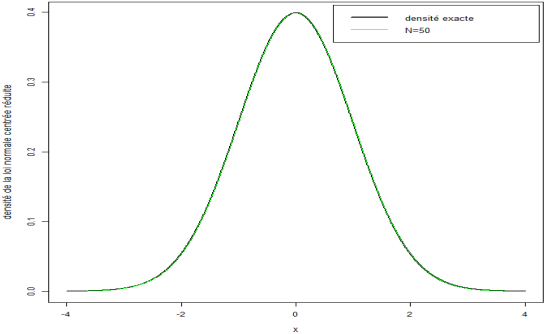
- No constraint in the demonstration of the COS method (F. Fang & C.W. Osterlee (2008)) imposed the use of a cosine extension.
- Therefore, the density of a process can also be approximated on the basis of a sine extension of a function ⇒ Alternative method called SIN method :

$$\boxed{f(x) \approx \frac{2}{b-a} \sum_{n=1}^{N} Im \left\{ \Phi\left(\frac{n\pi}{b-a}\right) exp\left(-i\frac{na\pi}{b-a}\right) \right\} sin\left(n\pi\frac{x-a}{b-a}\right)}$$

- In what follows
  - Comparison with the COS method for the approximation of common density functions
  - Comparison with the COS method for retrieving price and option Greeks
  - **New approach to GMWB valuation**

Approximation of density functions by the SIN method

- The standard normal distribution

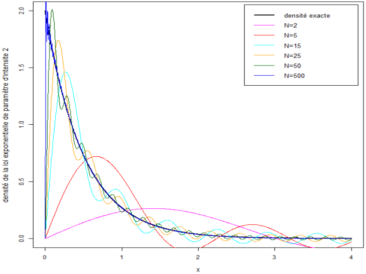


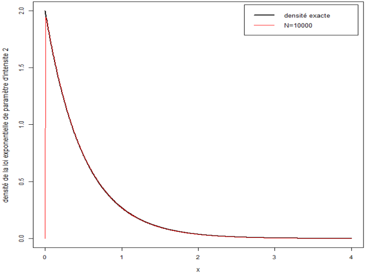
		N=5	N=25	N=50
max.errors	COS	0.23932	0.00348	1.092×10^{-7}
	SIN	0.21105	0.00254	6.21×10^{-8}

$\Rightarrow$ Better approximation by the SIN method.

Approximation of density functions by the SIN method

- The exponential distribution





Large oscillations close to $x = 0$.

Comparison of maximum errors for values of $x > 0.4$:

		N=5	N=25	N=50	N=500
max. errors	COS	0.24754	0.02950	0.00967	0.00012
	SIN	0.98995	0.33102	0.19000	0.01825

$\Rightarrow$ Better approximation by the COS method, but...

Valuation of options using the SIN method

- Price of a European call

	N=2	N=15	N=50	N=100
COS	14.98964	3.10188	3.10369	3.10369
SIN	13.60628	3.05657	3.10369	3.10369

Approximation of the price of a European call of parameters $S_0 = 100$, $r = 2\%$, $T = 0.1$, $\sigma = 0.15$ and $K = 100$ by the COS and SIN methods as a function of the parameter N (Black-Scholes value = 3.02757).

Same behavior, but shorter calculation time for the SIN method.
Same conclusions for a put option.

$\Rightarrow$ Therefore, the SIN method seems preferable to the COS method for option valuation problems.

Approximation of the Greeks

- Value of a European call option's Δ

		N=5	N=15	N=25	N=50	N=100	B-S
K=75	COS	0.50158	0.97163	0.98257	0.98260	0.98260	0.95433
	SIN	-0.16299	0.88995	0.98226	0.98260	0.98260	
K=100	COS	0.23766	0.66221	0.64962	0.64969	0.64969	0.59871
	SIN	-0.46368	0.74250	0.64909	0.64969	0.64969	
K=125	COS	0.31407	0.23927	0.22639	0.22641	0.22641	0.19332
	SIN	0.55929	0.09910	0.22618	0.22641	0.22641	

Several remarks:

- The 2 methods converge to identical values and stabilize from $N = 50$.
- For $K = 75{,}100$ and 125, the relative errors are 2.96%, 8.51% and 17.11%. $\Rightarrow$ Maximum errors for OTM options.
- Similar behaviour for put options

$\Rightarrow$ The 2 methods are equivalent to approximate the delta of an option

Approximation of the Greeks

- Value of a European call option's Γ

		N=5	N=15	N=25	N=50	N=100	B-S
K=75	COS	0.01493	0.00367	0.00410	0.00411	0.00411	0.00480
	SIN	0.04031	0.00246	0.00402	0.00411	0.00411	
K=100	COS	0.00220	0.01977	0.01913	0.01914	0.01914	0.01933
	SIN	-0.02376	0.02616	0.01908	0.01914	0.01914	
K=125	COS	-0.00841	0.01562	0.01520	0.01518	0.01518	0.01371
	SIN	-0.06122	0.01308	0.01534	0.01518	0.01518	

Several remarks:

- Convergence to similar values, and stable results from $N = 50$.
- Identical results for the gamma of put options with the same characteristics.
- For $K = 75, 100$ and 125, the relative errors are 14.37%, 0.98% and 10.72%, respectively. $\Rightarrow$ Negligible error for ATM options.

$\Rightarrow$ The 2 methods are equivalent to approximate the gamma of an option

The SIN method for valuing GMWBs

- The SIN method makes it possible to derive a recursive algorithm for the valuation of GMWBs :

$$V_{t_i}(W_{t_i}, A_{t_i}) = \sup_{\pi} \left[C(\theta_{t_i}) + e^{-r(t_{i+1}-t_i)} \left(\frac{2}{b-a} \right) \sum_{n=1}^{M} Im \left\{ \Phi^Q \left(\frac{n\pi}{b-a} \right) e^{-i\frac{na\pi}{b-a}} \right\} U_n^{SIN}(W_{t_i^+}, A_{t_i^+}) \right]$$

$$U_n^{SIN}\left(W_{t_{N-1}^+}, A_{t_{N-1}^+}\right) = A_{t_{N-1}^+} \psi_n^{SIN}(a, y^*) + W_{t_{N-1}^+} \chi_n^{SIN}(y^*, b)$$

$$U_n^{SIN}(W_{t_i^+}, A_{t_i^+}) = \int_a^b V_{t_{i+1}} \left(\max \left[W_{t_i^+}, 0 \right] e^y, A_{t_i^+} \right) \sin \left(n\pi \frac{y-a}{b-a} \right) dy$$

- The results will be compared with those of Bacinello et al. (2014) (based on FFT) and Alonso-Garcia et al. (2017) (based on the COS method)

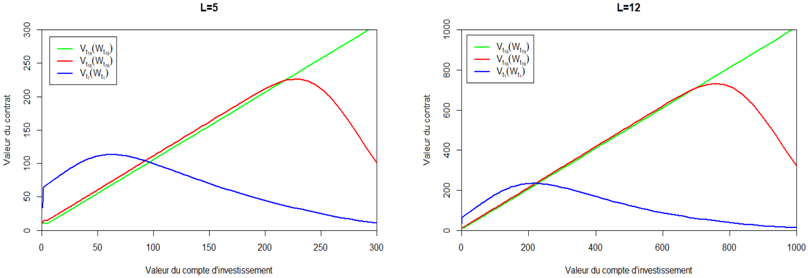
The SIN method for valuing GMWBs

- Choice of L parameter
  - The SIN method restricts the infinite integration interval to an interval [a,b] $\propto L$.
  - A value of L that is too large may increase the calculation time, a value that is too small makes the results inaccurate.

			J=20	J=80	J=400	J=1600	J=3200
L=5	GBM	COS	103.66	91.53	91.82	91.86	91.86
		SIN	115.99	91.15	91.52	91.53	91.53
	VG	COS	74.39	92.60	92.97	93.03	93.03
		SIN	61.80	92.44	92.66	92.66	92.66
L=12	GBM	COS	220.79	99.23	99.37	99.36	99.35
		SIN	134.42	98.87	99.24	99.26	99.27
	VG	COS	770.23	104.51	101.55	100.56	100.38
		SIN	64.28	106.69	106.91	106.91	106.91

Imperfect convergence of V_0 to 100, even for $L = 12$.

The SIN method for valuing GMWBs



- For W too big, the contract value drops sharply.
  - Problem : It is not possible to consider all the returns $y \in [a, b]$ when W becomes too large.
  - Solution : Increase the value of W_{max} of the discretization grid of the algorithm.
- By adopting this solution, the value of $L = 5$ becomes sufficient to have a satisfactory approximation.

The SIN method for valuing GMWBs

- Comparison of results for the VG process

		N=16	N=32	N=64	N=128
$J = 25$	COS	83.35	62.01	61.88	61.88
	SIN	96.91	102.60	102.54	102.54
$J = 50$	COS	99.51	99.88	99.28	99.28
	SIN	99.76	99.56	99.48	99.48
$J = 250$	COS	100.05	100.02	100.01	100.01
	SIN	100.74	100.16	100.15	100.15
$J = 1000$	COS	100.04	100.01	100.01	100.01
	SIN	100.77	100.15	100.15	100.15
Bac. (FFT)		110.51	102.39	100.5	100.07

Remarks:

1. For J large enough, the COS method is constantly more accurate than FFT.
2. The SIN method is more accurate than FFT for N small, but this result is reversed when N increases.
3. In the case of the GBM, the COS and SIN methods are constantly more accurate than the FFT.

Conclusions

- **The COS method makes it possible to approximate the density functions of classical distributions, as well as the price and Greeks of options** in a precise way, with certain limits.
- The accuracy and speed of convergence of the method are **significantly better than classical Fourier transform methods for finding density functions of Levy processes**.
- **The SIN method**, an alternative to the COS method, has also **appeared to be preferable to the COS method in many issues** (e.g. normal density function).
- **The results provided by these two methods** were also **superior to those provided by the FFT for the treatment of GMWBs** in the majority of cases.
- Nevertheless, the main limitation of these 2 methods comes from the choice of the $[a, b]$ interval, and future studies could focus on the determination of an optimal interval.

Thank you for your attention